Integral

If the function $F(x)$ is an antiderivative of $f(x)$ then the expression $F(x)+C$ where C is an arbitrary constant, is called the indefinite integral of $f(x)$ and we write

$$\int f(x)\,dx = F(x) + C$$

The integral of the power function

If u is a function of x, then

1. $\int u^n\,du = \frac{u^{n+1}}{n+1} + c \;; n \in \mathbb{R}\;,\;\; n \neq -1$

2. $\int \frac{du}{u} = \ln|u| + c$

Examples

1. $\int \sqrt{3x+1}\,dx = \frac{1}{3}\int 3\,(3x+1)^{\frac{1}{2}}dx = \frac{1}{3}(3x+1)^{\frac{3}{2}} \times \frac{2}{3} + c = \frac{2}{9}(3x+1)^{\frac{3}{2}} + c$

2. $\int \frac{xdx}{\sqrt{x^2-3}} = \frac{1}{2}\int 2x(x^2-3)^{-1/2}dx = \frac{1}{2}(x^2-3)^{1/2} \times 2 + c = \sqrt{x^2-3} + c$

3. $\int \frac{xdx}{x^2-3} = \frac{1}{2}\ln|x^2-3| + c$

The integrals of trigonometric functions

If u is a function of x, then

1. $\int \sin u\,du = -\cos u + c$

2. $\int \cos u\,du = \sin u + c$

3. $\int \sec^2 u\,du = \tan u + c$

4. $\int \csc^2 u\,du = -\cot u + c$

5. $\int \sec u \tan u\,du = \sec u + c$

6. $\int \csc u \cot u\,du = -\csc u + c$

7. $\int \tan u\,du = -\ln|\cos u| + c = \ln|\sec u| + c$

8. $\int \cot u\,du = \ln|\sin u| + c = -\ln|\csc u| + c$

Examples

4. $\int \sqrt{1+\cos 2x}\,dx = \int \sqrt{2\cos^2 x}\,dx = \int \sqrt{2}\cos x\,dx = \sqrt{2}\sin x + c$

5. $\int \frac{\sin 2x}{\sqrt{1+\cos 2x}}dx = \int (1+\cos 2x)^{-1/2}\sin 2x\,dx = \sqrt{1+\cos 2x} + c$

6. $\int \frac{\sin 2x}{1+\cos 2x}dx = \frac{1}{2}\int \frac{2\sin 2x}{1+\cos 2x}dx = \frac{1}{2}\ln(\cos 2x) + c$

The integrals of exponential functions

If u is a function of x, then

1. $\int a^u du = \frac{a^u}{\ln a} + c \;\; ; \;\; a > 0, a \neq 1$ 2. $\int e^u\,du = e^u + c$

Examples

7. $\int e^{3x}\,dx = \frac{1}{3}e^{3x} + c$

8. $\int (2e^{-2x} + 3x^2)\,dx = -e^{-2x} + x^3 + c$

The integrals concerning the inverse trigonometric functions

1. $\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + c_1 = -\cos^{-1}\frac{u}{a} + c_2$

2. $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + c_1 = -\frac{1}{a}\cot^{-1}\frac{u}{a} + c_2$

3. $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + c_1 = -\frac{1}{a}\csc^{-1}\frac{u}{a} + c_2$

Examples

9. $\int \frac{dx}{\sqrt{2-x^2}} = \sin^{-1}\frac{x}{\sqrt{2}} + c$

10. $\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{x^2+2x+1+1}$

$$= \int \frac{dx}{(x+1)^2+1} = \tan^{-1}(x+1) + c$$

$$11. \int \frac{dx}{x\sqrt{x^2-3}} = \frac{1}{\sqrt{3}} \sec^{-1} \frac{x}{\sqrt{3}} + c$$

$$12. \int \frac{2x+7}{x^2+4x+5} dx = \int \frac{2x+4+3}{x^2+4x+5} dx$$

$$= \int \frac{2x+4}{x^2+4x+5} dx + \int \frac{3}{x^2+4x+4+1} dx$$

$$= \ln|x^2+4x+5| + \int \frac{3}{(x+2)^2+1} dx$$

$$= \ln|x^2+4x+5| + 3\tan^{-1}(x+2) + c$$

Exercises

Evaluate the following integrals

1. $\int (x+\sqrt{x}) dx$
2. $\int \sqrt[3]{3x-2}\, dx$
3. $\int \frac{x dx}{\sqrt{1-4x^2}}$
4. $\int \frac{x dx}{x^2+3}$
5. $\int \frac{dx}{\sqrt{1-4x^2}}$
6. $\int \frac{dx}{x^2+3}$
7. $\int \sqrt{1-\cos 4x}\, dx$
8. $\int \frac{(x-1)^2}{x\sqrt{x}} dx$
9. $\int \frac{dx}{x^2-2x+5}$
10. $\int \frac{(2x-5)dx}{x^2-4x+5}$

Methods of integration

I- Integration by parts

Sometimes we can recognize the differential to be integrated as a product of a function which is easily differentiated and a differential which is easily integrated.

For example, if the problem is to find the integral

$$\int x^2 \cos x \, dx$$

We have two way to do this:

1- Tabular integration by parts

Which is used for

1) the products of polynomials and sine function, $x^n \sin bx$.

2) the products of polynomials and cosine function, $x^n \cos bx$.

3) the products of polynomials and exponential function, $x^n e^{ax}$.

In any of these three cases we choose the polynomial as u and the product of sine function and dx (cosine or exponential function and dx – respectively) as dv.

Example1: Evaluate

$$1. \int x^3 \cos x \, dx \qquad 2. \int x^2 e^{2x} \, dx$$

	Derivatives of u		Integrals of v
$1. \int x^3 \cos x \, dx$	x^3	$+$	$\cos x$
	$3x^2$	$-$	$\sin x$
	$6x$	$+$	$-\cos x$
	6	$-$	$-\sin x$
	0		$\cos x$

$$\int x^3 \cos x \, dx = x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + c$$

2. $\int x^2 e^{2x}\,dx$

$D. of\ x^2$		$I. of\ e^{2x}$
x^2	$+$	e^{2x}
$2x$	$-$	$\frac{1}{2}e^{2x}$
2	$+$	$\frac{1}{4}e^{2x}$
0		$\frac{1}{8}e^{2x}$

$$\int x^2 e^{2x}\,dx = \frac{x^2}{2}e^{2x} - \frac{2x}{4}e^{2x} + \frac{2}{8}e^{2x} + c = \frac{e^{2x}}{2}\left(x^2 - x + \frac{1}{2}\right) + c$$

2- Integration by using the formula

$$\int u\,dv = u\,v - \int v\,du$$

Which is used for inverse trigonometric functions and logarithm functions.

Example2: Evaluate

1. $\int \ln x\,dx$ 2. $\int \sin^{-1} 2x\,dx$

3. $\int_1^4 \sec^{-1}\sqrt{x}\,dx$

1. $u = \ln x$ and $dv = dx$

$du = \frac{dx}{x}$ and $v = x$

$$\int u\,dv = u\,v - \int v\,du$$

$$\int \ln x\,dx = x\ln x - \int x \times \frac{dx}{x}$$

$$= x\ln x - \int dx = x\ln x - x + c$$

2. $u = \sin^{-1} 2x$ and $dv = dx$

$$du = \frac{2dx}{\sqrt{1-4x^2}} \quad \text{and} \quad v = x$$

$$\int u\, dv = u\, v - \int v\, du$$

$$\int \sin^{-1} 2x\, dx = x \sin^{-1} 2x - \int \frac{2x dx}{\sqrt{1-4x^2}}$$

$$= x \sin^{-1} 2x + \frac{1}{2}\sqrt{1-4x^2} + c$$

3. $u = \sec^{-1}\sqrt{x}$ and $dv = dx$

$$du = \frac{dx}{\sqrt{x}\sqrt{x-1}} \times \frac{1}{2\sqrt{x}} = \frac{dx}{2x\sqrt{x-1}} \quad \text{and} \quad v = x$$

$$\therefore \int_1^4 \sec^{-1}\sqrt{x}\, dx = x\sec^{-1}\sqrt{x}\Big|_1^4 - \int_1^4 \frac{x dx}{2x\sqrt{x-1}}$$

$$= 4\sec^{-1} 2 - \sec^{-1} 1 - \int_1^4 \frac{dx}{2\sqrt{x-1}}$$

$$= 4 \times \frac{\pi}{3} - 0 - \frac{1}{2}\sqrt{x-1} \times 2\Big|_1^4$$

$$= \frac{4\pi}{3} - (\sqrt{3} - 0) = \frac{4\pi}{3} - \sqrt{3}$$

Exercises

1. $\int x \ln x\, dx$

2. $\int x^3 e^{-3x}\, dx$

3. $\int \tan^{-1} x\, dx$

4. $\int_0^1 \frac{\sin^{-1}\sqrt{x}}{\sqrt{x}}\, dx$

5. $\int (x^2 + 3x) \sin 2x dx$

6. $\int_0^{1/\sqrt{2}} 2x \sin^{-1}(x^2) dx$

II- Integration by the mothed of partial fractions

This method is based on the simple concept of adding fractions by getting a common denominator. For example,

$$\frac{1}{x-1}+\frac{2}{x+5}=\frac{x+5+2(x-1)}{(x-1)(x+5)}=\frac{3x+3}{(x-1)(x+5)}$$

so that we can now say that a partial fractions decomposition is

$$\frac{3x+3}{(x-1)(x+5)}=\frac{1}{x-1}+\frac{2}{x+5}$$

Partial fractions can only be done if the degree of the numerator is strictly less than the degree of the denominator.

For each factor in the denominator we can use the following table to determine the terms we pick up in the partial fraction decomposition.

Factor in denominator	Term in partial fraction decomposition
$(x+a_1)(x+a_2)\cdots(x+a_n)$	$\frac{A_1}{(x+a_1)}+\frac{A_2}{(x+a_2)}+\cdots+\frac{A_n}{(x+a_n)}$
Repeated roots: $(x+a)^n$	$\frac{A_1}{(x+a)}+\frac{A_2}{(x+a)^2}+\cdots+\frac{A_n}{(x+a)^n}$
$(x^2+a_1)(x^2+a_2)\cdots(x^2+a_n)$	$\frac{A_1x+B_1}{(x^2+a_1)}+\frac{A_2x+B_2}{(x^2+a_2)}+\cdots+\frac{A_nx+B_n}{(x^2+a_n)}$

There are several methods for determining the coefficients for each term and we will go over each of those in the following examples.

Example: Evaluate the following integral.

1. $\int \frac{3x-4}{x^2-x-6}dx$

2. $\int \frac{x^2-4x+9}{(x+2)^3}dx$

3. $\int \frac{x^2+2x-5}{x^4+4x^2+3}dx$

1. $\int \frac{3x-4}{x^2-x-6}dx$

$$\frac{3x-4}{(x-3)(x+2)} = \frac{A}{(x-3)} + \frac{B}{(x+2)}$$

To find A : multiply both sides by $(x-3)$ and plug in $x=3$

$$\frac{3x-4}{(x+2)} = A + \frac{B(x-3)}{(x+2)} \quad ⇨ \quad A = \frac{3\times 3-4}{(3+2)} = 1$$

To find B : multiply both sides by $(x+2)$ and plug in $x=-2$

$$\frac{3x-4}{(x-3)} = \frac{A(x+2)}{(x-3)} + B \quad ⇨ \quad B = \frac{3\times(-2)-4}{(-2-3)} = 2$$

$$\int \frac{3x-4}{x^2-x-6}dx = \int \left(\frac{1}{(x-3)} + \frac{2}{(x+2)}\right)dx$$

$$= \ln|x-3| + 2\ln|x=2| + c$$

2. $\int \frac{x^2-4x+9}{(x+2)^3}dx$□

$$\frac{x^2-4x+9}{(x+2)^3} = \frac{A}{(x+2)^3} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)}$$□

To find A : multiply both sides by $(x+2)^3$ and plug in $x=-2$

$$x^2-4x+9 = A + B(x+2) + C(x+2)^2$$□

$$(-2)^2-4(-2)+9 = A \quad ⇨ \quad A = 21$$□

To find B : differentiate once and plug in $x=-2$

$$2x-4 = B + 2C(x+2)$$

$$2(-2)-4 = B \quad ⇨ \quad B = -8$$

To find C : differentiate again and plug in $x=-2$

$$2 = 2C \quad ⇨ \quad C = 1$$

$$\int \frac{x^2-4x+9}{(x+2)^3}dx = \int \left(\frac{21}{(x+2)^3} + \frac{-8}{(x+2)^2} + \frac{1}{(x+2)}\right)dx$$

$$= \frac{-21}{2(x+2)^2} + \frac{8}{(x+2)} + \ln|x=2| + c$$

3. $\int \frac{x^2+2x-5}{x^4+4x^2+3}dx$

$$\frac{x^2+2x-5}{(x^2+3)(x^2+1)} = \frac{Ax+B}{(x^2+3)} + \frac{Cx+D}{(x^2+1)}$$

$$= \frac{(Ax+B)(x^2+1)+(Cx+D)(x^2+3)}{(x^2+3)(x^2+1)}$$

$$x^2+2x-5 = Ax^3+Bx^2+Ax+B+Cx^3+Dx^2+3Cx+3D$$

Coefficient of x^3: $A+C=0$

Coefficient of x^2: $B+D=1$

Coefficient of x: $A+3C=2$

Constant term : $B+3D=-5$

This gives us $C=1$, $A=-1$, $D=-3$ and $B=4$

$$\int \frac{x^2+2x-5}{x^4+4x^2+3}dx = \int \left(\frac{-x+4}{(x^2+3)} + \frac{x-3}{(x^2+1)}\right)dx$$

$$= -\int \frac{x}{(x^2+3)}dx + \int \frac{4}{(x^2+3)}dx + \int \frac{x}{(x^2+1)}dx - \int \frac{3}{(x^2+1)}dx$$

$$= -\frac{1}{2}\ln(x^2+3) + \frac{4}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + \frac{1}{2}\ln(x^2+1) + 3\tan^{-1}x + c$$

Exercises

Evaluate the following integral

1. $\int \frac{3x+5}{x^2-2x-15}dx$

2. $\int \frac{2x^2+3x+2}{(x+1)^3}dx$

3. $\int \frac{x^2-3x+7}{x^4+5x^2+4}dx$

Multiple Integrals

The multiple integral is a generalization of the definite integral to functions of more than one real variable. Integrals of a function of two variables are called double integrals, and integrals of a function of three variables are called triple integrals.

Double Integrals

The expression $\int_c^d \int_a^b f(x,y)dxdy$ is called double integral and indicates that:

1. $f(x,y)$ is first integrated with respect to x (regarding y as being constant) between the limits $x = a$ and $x = b$.

2. the result is then integrated with respect to y between the limits $y = c$ and $y = d$.

Example 1: Evaluate $\int_1^2 \int_2^4 (x+2y)dxdy$

$$\int_1^2 \int_2^4 (x+2y)dxdy = \int_1^2 \frac{x^2}{2} + 2yx\Big|_2^4 dy = \int_1^2 (8+8y-2-4y)\,dy$$

$$= \int_1^2 (4y+6)\,dy = 2y^2 + 6y\Big|_1^2 = 8+12-2-6 = 12$$

Example 2: Evaluate $\int_0^\pi \int_0^{\sin x} y\,dydx$

$$\int_0^\pi \int_0^{\sin x} y\,dydx = \int_0^\pi \frac{y^2}{2}\Big|_0^{\sin x} dx = \int_0^\pi \frac{\sin^2 x}{2} dx = \int_0^\pi \frac{1-\cos 2x}{4} dx$$

$$= \frac{1}{4}x - \frac{1}{8}\sin 2x\Big|_0^\pi = \frac{\pi}{4}$$

Example 3: Evaluate $\int_1^2 \int_y^{y^2} dxdy$

$$\int_1^2 \int_y^{y^2} dxdy = \int_1^2 x\Big|_y^{y^2} dy = \int_1^2 (y^2 - y)\, dy = \frac{y^3}{3} - \frac{y^2}{2}\Big|_1^2 = \frac{5}{6}$$

Example 4: Evaluate $\int_0^1 \int_0^x \sqrt{1-x^2}\, dydx$

$$\int_0^1 \int_0^x \sqrt{1-x^2}\, dydx = \int_0^1 y\Big|_0^x \sqrt{1-x^2} dx = \int_0^1 x\sqrt{1-x^2}\, dx = -\frac{1}{3}(1-x^2)^{3/2}\Big|_0^1 = \frac{1}{3}$$

Triple Integrals

It will come as no surprise that we can also do triple integrals

Example 5: Evaluate $\int_0^2 \int_{-1}^1 \int_0^1 (2x - y + z) dxdydz$

$$\int_0^2 \int_{-1}^1 \int_0^1 (2x - y + z) dxdydz = \int_0^2 \int_{-1}^1 (x^2 - yx + zx)\Big|_0^1 dydz = \int_0^2 \int_{-1}^1 (1 - y + z) dydz$$

$$= \int_0^2 \left(y - \frac{y^2}{2} + zy\right)\Big|_{-1}^1 dz = \int_0^2 \left(1 - \frac{1}{2} + z - \left(-1 - \frac{1}{2} - z\right)\right) dz$$

$$= \int_0^2 2zdz = z^2 \Big|_0^2 = 4$$

Example 6: Evaluate $\int_0^\pi \int_0^\pi \int_0^3 x^2 \sin\theta\, dxd\theta d\emptyset$

$$\int_0^\pi \int_0^\pi \int_0^3 x^2 \sin\theta\, dxd\theta d\emptyset = \int_0^\pi \int_0^\pi \left(\frac{x^3}{3}\Big|_0^3\right) \sin\theta\, d\theta d\emptyset = \int_0^\pi \int_0^\pi 9 \sin\theta\, d\theta d\emptyset$$

$$= \int_0^\pi -9\cos\theta\Big|_0^\pi d\emptyset = \int_0^\pi -9(-1-1) d\emptyset = \int_0^\pi 18 d\emptyset = 18\emptyset\Big|_0^\pi = 18\pi$$

Exercises

Evaluate

1. $\int_0^2 \int_1^{e^x} dydx$

2. $\int_0^1 \int_{\sqrt{y}}^1 dxdy$

3. $\int_{-2}^1 \int_{x^2+4x}^{3x+2} dydx$

4. $\int_0^1 \int_{\sqrt{y}}^{2-\sqrt{y}} xydxdy$

5. $\int_0^1 \int_0^{y^2} \sqrt{y^3+3}\, dxdy$

6. $\int_0^2 \int_0^1 \int_0^1 (x^2+y^2+z^2)dxdydz$

7. $\int_0^2 \int_1^3 \int_1^2 xy^2zdxdydz$

المشتقات الجزئية Partial Derivatives

الدوال في متغيرين : يُسمى المتغير $z = f(x,y)$ دالة بالمتغيرين x و y اذا كان لكل زوج (x,y) توجد قيمة واحدة للمتغير z.

مثلا اذا كان $f(x,y) = x^2y - 3xy$ فان $f(2,-1) = 2^2 \times (-1) - 3 \times 2 \times (-1) = 2$

المشتقات الجزئية

المشتقة الجزئية للدالة $f(x,y)$ بالنسبة للمتغير x هي نفس المشتقة الاعتيادية للدالة $f(x,y)$ بالنسبة للمتغير x وذلك باعتبار y ثابت وتكتب $\frac{\partial f}{\partial x}$ أو f_x و المشتقة الجزئية للدالة $f(x,y)$ بالنسبة للمتغير y هي نفس المشتقة الاعتيادية للدالة $f(x,y)$ بالنسبة للمتغير y وذلك باعتبار x ثابت وتكتب $\frac{\partial f}{\partial y}$ أو f_y .

فمثلا : اذا كانت $f(x,y) = e^{2x} \cos y$ فان $f_x = 2e^{2x} \cos y$ و $f_y = -e^{2x} \sin y$

المشتقات الجزئية من رتبة أعلى

اذا كانت الدالة $f(x,y)$ لها مشتقات جزئية فان $\frac{\partial f}{\partial x}$ و $\frac{\partial f}{\partial y}$ هي نفسها دوال ويمكن ان يكون لها مشتقات جزئية ، هذه المشتقات الثانية تأخذ الرموز :

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} \qquad \frac{\partial^2 f}{\partial y^2} = f_{yy} \qquad \frac{\partial^2 f}{\partial x \partial y} = f_{xy} \qquad \frac{\partial^2 f}{\partial y \partial x} = f_{yx}$$

مثال (١): جد : f_{xx} ، f_{yy} ، f_{yx} و f_{xy} للدوال التالية :

1. $f(x,y) = x^2 + xy^2$

2. $f(x,y) = \ln(2x+2y) + \tan(2x-2y)$

الحل :

1. $f_x = 2x + y^2 \rightarrow f_{xx} = 2 \quad , \quad f_{xy} = 2y$

$f_y = 2xy \rightarrow f_{yy} = 2x \quad , \quad f_{yx} = 2y$

2. $$f_x = \frac{2}{2x+2y} + 2\sec^2(2x-2y) = \frac{1}{x+y} + 2\sec^2(2x-2y)$$

$$f_{xx} = \frac{-1}{(x+y)^2} + 8\sec^2(2x-2y)\,\tan(2x-2y)$$

$$f_{xy} = \frac{-1}{(x+y)^2} - 8\sec^2(2x-2y)\,\tan(2x-2y)$$

$$f_y = \frac{2}{2x+2y} - 2\sec^2(2x-2y) = \frac{1}{x+y} - 2\sec^2(2x-2y)$$

$$f_{yy} = \frac{-1}{(x+y)^2} + 8\sec^2(2x-2y)\,\tan(2x-2y)$$

$$f_{xy} = \frac{-1}{(x+y)^2} - 8\sec^2(2x-2y)\,\tan(2x-2y)$$

معادلة لابلاس : لتكن الدالة $f(x,y)$ قابلة للاشتقاق فمعادلة لابلاس هي $f_{xx} + f_{yy} = 0$

مثال (٢): بين ان الدالة $f(x,y) = e^{-2y}\cos 2x$ تحقق معادلة لابلاس

الحل : $f_x = -2e^{-2y}\sin 2x \quad \rightarrow \quad f_{xx} = -4e^{-2y}\cos 2x$

$f_y = -2e^{-2y}\cos 2x \quad \rightarrow \quad f_{yy} = 4e^{-2y}\cos 2x$

$f_{xx} + f_{yy} = -4e^{-2y}\cos 2x + 4e^{-2y}\cos 2x = 0$

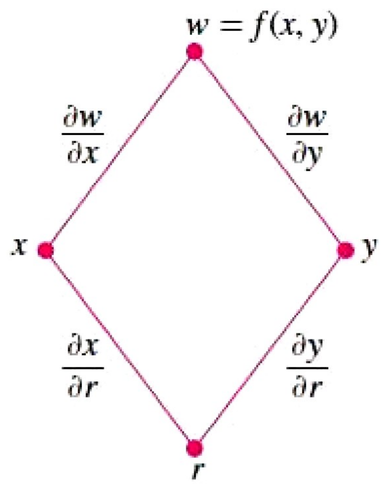
وعليه فان الدالة $f(x,y) = e^{-2y}\cos 2x$ تحقق معادلة لابلاس

مشتقات الدوال المؤلفة : (قاعدة السلسلة)

اذا كانت $w = f(x,y)$ حيث $x = g(r,s)$ و $y = h(r,s)$ فان :

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \quad and \quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

Chain Rule



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

مثال (٣): جد $\partial w/\partial s$ و $\partial w/\partial r$ بدلالة s و r اذا علمت ان $w = x^2 + y^2, x = r - s, y = r + s$

الحل: $w_x = 2x = 2r - 2s \;,\; w_y = 2y = 2r + 2s$

$\partial x/\partial r = 1 \;,\quad \partial x/\partial s = -1 \;,\quad \partial y/\partial r = 1 \;,\quad \partial y/\partial s = 1$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} = (2r - 2s)\times(-1) + (2r + 2s)\times 1 = 4s$$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} = (2r - 2s)\times 1 + (2r + 2s)\times 1 = 4r$$

تمارين

جد f_{xx} , f_{yy} , f_{yx} و f_{xy} للدوال التالية :

1. $f(x,y) = x^2 \tan^{-1}\dfrac{y}{x}$
2. $f(x,y) = \ln(xy) + \tan(xy)$
3. $f(x,y) = x^2 e^x \sin y + 3x - 2\cos(x+2y)$
4. $f(x,y) = e^{xy} + \tan^{-1}(xy)$

اثبت ان $\dfrac{\partial^2 w}{\partial y^2} = \dfrac{\partial^2 w}{\partial x^2}$ اذا كان :

5. $w = \cos(x+y) + \sin(x-y)$
6. $w = \cos(2x-2y) + e^x \sinh y$

جد $\partial z/\partial v$ و $\partial z/\partial u$ عند النقطة المعطاة

7. $z = e^x \ln y \quad , \quad x = \ln(u\cos v) \quad , \quad y = u\sin v \quad , \quad (u,v) = (2, \pi/4)$
8. $z = \tan^{-1}(x/y) \quad , \quad x = u\cos v \quad , \quad y = u\sin v \quad , \quad (u,v) = (1.3, \pi/6)$

بين ان الدوال التالية تحقق معادلة لابلاس

9. $f(x,y) = \ln\sqrt{x^2+y^2}$
10. $f(x,y) = e^{-3x}\sin 3y$

11. اذا كانت $u = ax + by \quad , \quad w = u^3 + \tanh u + \cos u$ حيث a, b ثوابت فاثبت ان

$$a\frac{\partial w}{\partial y} = b\frac{\partial w}{\partial x}$$